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Abstract

We present a generalization of Permutative logic (PL) [1] which is a non-commutative variant of Linear logic suggested by some topological investigations on the geometry of linear proofs. The original logical status based on a variety-presentation framework is simplified by extending the notion of q-permutation to the one of pq-permutation [7]. Whereas PL is limited to orientable structures, we characterize the whole range of topological surfaces, orientable as well as non-orientable. The system we obtain is a surface calculus that enjoys both cut elimination and focussing properties and comes with a natural phase semantics whenever explicit context is considered.

Among the different viewpoints considered for studying proofs of Linear Logic, let us recall that its graph-theoretical representations may be seen as topological objects and considered as surfaces on which usual proofs are drawn without crossing edges [2, 6, 5]. Following that interpretation, Gaubert [3] provided a way to *compute* surfaces. Moreover, based on the fact that the exchange rule may model topological operations, non-commutative variants of Multiplicative Linear logic (MLL) were developed: planar logic [5], the calculus of surfaces [3] and permutative logic (PL) [1]. In all these cases the conclusions of the proofs are drawn on disjoint oriented circles, more precisely orientable surfaces with boundary. E.g. in PL the underlying structure is that of a permutation, which is a product of disjoint cycles, together with a natural number to express the number of tori, actually a topological invariant of the surface. The shape of such sequents is called a q-permutation and geometrically studied in [7] by one of the present authors. One of the main topological results given in [7] is related to Massey classification theorem [4]: any orientable surface, possibly with boundary, is homeomorphic either to a sphere or to a finite connected sum of tori, possibly with boundary.

We consider in this paper a generalization to surfaces orientable or not. Massey theorem states in that case that it may be homeomorphic also to a finite connected sum of projective planes, possibly with boundary. For that purpose, we consider in our work pq-permutations which are simply obtained from q-permutations by replacing the single index q with an ordered couple (p, q) of positive integers for counting tori handles and projective planes. The shape of our sequents still integrates the topology of a surface and non-trivial exchange rules correspond to surface transformations, following what is done in PL but also in Melliès planar logic [5]. After presenting the logical system, we prove a few logical properties: it enjoys both cut elimination and the focussing property, as PL does. We give a phase semantics that is sound and complete with respect to the calculus. Though a phase semantics may seem a too elementary result, it allows us to tackle the problem of contextual structures. The aim of our work is to shed new light on the relationship between topology and logic.

1 sPL: A Sequent Calculus for Surfaces

Formulas of sPL are inductively built from a countable infinite set of atoms $\mathcal{A} = \{a, b, c, \dots, a^\perp, b^\perp, c^\perp, \dots\}$ and the two usual multiplicative connectives $⅋$ and $\otimes$, together with a unary bar operation $(\bar{\_})$ that models the inversion of the orientation:

$$F ::= F \in \mathcal{A} \mid \bar{F} \mid F_1 ⅋ F_2 \mid F_1 \otimes F_2$$

IDENTITY GROUP

$$\frac{}{\vdash^0_0 (A, A^\perp)}\ \text{ax.} \qquad \frac{\vdash^p_q \Sigma, (\Gamma, A) \qquad \vdash^{p'}_{q'} \Xi, (\Delta, A^\perp)}{\vdash^{p+p'}_{q+q'} \Sigma, \Xi, (\Gamma, \Delta)}\ \text{cut}$$

ORIENTABLE STRUCTURAL RULES

$$\frac{\vdash^p_q \Sigma, (\Gamma, \Delta)}{\vdash^p_q \Sigma, (\Gamma), (\Delta)}\ \text{cylinder} \qquad \frac{\vdash^p_q \Sigma, (\Gamma), (\Delta)}{\vdash^p_{q+1} \Sigma, (\Gamma, \Delta)}\ \text{torus} \qquad \frac{\vdash^p_q \Sigma}{\vdash^p_q \overline{\Sigma}}\ \text{invert}$$

NON-ORIENTABLE STRUCTURAL RULES

$$\frac{\vdash^p_q \Sigma, (\Gamma, \Delta)}{\vdash^{p+1}_q \Sigma, (\Gamma, \overline{\Delta})}\ \text{Möbius} \qquad \frac{\vdash^p_q \Sigma, (\Gamma), (\Delta)}{\vdash^{p+2}_q \Sigma, (\Gamma, \overline{\Delta})}\ \text{Klein}$$

LOGICAL RULES

$$\frac{\vdash^p_q \Sigma, (\Gamma, A, B)}{\vdash^p_q \Sigma, (\Gamma, A \parr B)}\ \parr \qquad \frac{\vdash^p_q \Sigma, (\Gamma, A) \qquad \vdash^{p'}_{q'} \Xi, (\Delta, B)}{\vdash^{p+p'}_{q+q'} \Sigma, \Xi, (\Gamma, A \otimes B, \Delta)}\ \otimes$$

Table 1: Sequent calculus for sPL

The negation is defined as usual by de Morgan duality and preserves the bar operation. A *sequent* is denoted $\vdash^p_q \Gamma$ where Γ is a multiset of cyclic sequences which are formulas separated by ',' within parenthesis, and p and q are integers with the intuition they denote a pq-permutation (p for projective planes). We write () for an empty cycle. Type derivations are built from the rules of table 1.

Remark that the key rules of divide and merge in PL are also in sPL as respectively cylinder and torus acting as orientable rules.

2 Cut Elimination and Focussing Property

Theorem 1 (cut-elimination)**.** *Any proof of a sPL sequent can be rewritten into a cut-free proof of the same sequent.*

A standard proof by case analysis need a particular attention for commutative conversions involving the Möbius or the Klein rule. This result can also be obtained as a consequence of *focalization.*

A focalized sequent calculus, called foc-sPL, may be defined with sequents of the form $\vdash^p_q \Gamma|\Sigma$ where Γ is the focus – a distinguished cyclic sequence – and Σ is a multiset of cyclic sequences of formulas separated by ';'. The cut rule acts only on focusses. The main ingredients are the following ones: a focus rule is added and Klein and torus rules are changed in the following way:

$$\frac{\vdash^p_q |(\Gamma); \Sigma}{\vdash^p_q \Gamma|\Sigma}\ \text{focus} \qquad \frac{\vdash^p_q \Gamma, \Lambda, \Delta|\Sigma}{\vdash^p_{q+1} \Gamma, \Delta, \Lambda|\Sigma}\ \text{torus'} \qquad \frac{\vdash^p_q \Gamma, \Lambda, \Delta|\Sigma}{\vdash^{p+2}_q \Gamma, \Delta, \overline{\Lambda}|\Sigma}\ \text{Klein'}$$

In such a presentation the defocus rule is simply a special case of the cylinder rule (with () neutral w.r.t. ';'). As it follows from topological considerations, structures of proofs in foc-sPL may be normalized in such a way that cylinder applications arrive only at the end of a proof construction. Such proofs are called *maximally focalized* and it is then possible to prove a cut-elimination property on them.

Proposition 2 (Maximal Focalization)**.** *A sequent is provable in foc-sPL if and only if there exists a proof such that cylinder rules are applied only at the end. Moreover cuts in a maximally focalized proof in foc-sPL may be eliminated.*

Sketch. As the cut rule is applied to focalized formulas and that cut and cylinder rules commute, one may consider that cuts are applied to sequents of the form $\vdash^p_q \Gamma|$. The rest is done by case analysis. □

We are finally able to prove that provability in sPL and foc-sPL are equivalent, hence a cut-elimination theorem for sPL follows.

Proposition 3 (Focussing property). *A sequent is provable in sPL if and only if it is provable in foc-sPL.*

3 Phase Semantic

A *phase space* is provided that is proved to be *complete and valid* with respect to the calculus. This should be considered as a first step towards a better understanding of the calculus and its relation to geometry. In fact, this is not at all obvious if we notice that there is not yet satisfying proof semantics for non-commutative logic (NL) even though its phase space has been given together with its sequent calculus (by Ruet in his 1997'thesis). What is the main difficulty when turning to a calculus of surfaces? Or equivalently what makes NL an easier situation? The orientation has to be taken into account, more than that the context cannot be neglected. In NL, the non-commutative structure is an order variety. Hence a formula on which an operation is applied may be 'extracted' from its context: the structure of the semantics is close to what is required with Linear Logic. This is no more true in the calculus of surfaces as *see-saw* structural rules are not valid: one is required to deal explicitly with the context.

For that purpose, a *support* phase space $\texttt{Supp(}M\texttt{)}$ interpreting formulas is embedded into a *context* phase space $\texttt{Con(}M\texttt{)}$ interpreting sequents. The two phase spaces are defined from an associative monoid and give rise to two closure operations $\perp$ and $\dagger$ in such a way that the fundamental proposition is provable:

Proposition 4. *Let $\mathcal{M} = (M, \star, 1)$ be a (not necessarily commutative) associative monoid with neutral element 1, let $F, G \subset \mathcal{M}$,*

$$(F \star G^{\perp\perp})^{\perp\perp} = (F \star G)^{\perp\perp}$$

$$(F \star G^{\dagger\dagger})^{\dagger\dagger} = (F \star G)^{\dagger\dagger}$$

We consider as usual that a *fact* is a subset A of the support phase space such that $A^{\perp\perp} = A$. Operations are defined on facts:

$$\begin{array}{llll} A \otimes B \stackrel{\text{def}}{=} (A \star B)^{\perp\perp} & A \parr B \stackrel{\text{def}}{=} (B^{\perp} \star A^{\perp})^{\perp} & 1 \stackrel{\text{def}}{=} \perp^{\perp} & \perp \text{ is given} \\ A \oplus B \stackrel{\text{def}}{=} (A \cup B)^{\perp\perp} & A \,\&\, B \stackrel{\text{def}}{=} A \cap B \top \stackrel{\text{def}}{=} \mathcal{M} & \mathbf{0} \stackrel{\text{def}}{=} \top^{\perp} & \end{array}$$

Although the general lines for proving soundness and validity of the model are standard, their proofs are more complex as they require to consider explicitly the context.

4 Principal line of current work: Relaxation

Relaxation is the binary relation induced by structural transformations – *divide*, *merge*, *Möbius* and *Klein* – on the set of *pq*-permutations. We write $\alpha \prec \beta$, β *relaxes* α, for meaning that the *pq*-permutation α can be rewritten into β through a suitable series of applications of structural rules. Since each structural rule increases the topological genus of the transformed surface, relaxation turns out to impose a partial order on the set of *pq*-permutations. We pose the problem of providing an algorithm for the decision of relaxation, namely an effective procedure able to answer to the question '$\alpha \prec \beta$?' being given two *pq*-permutations α and β.

Two parallel solutions have been already afforded in case of *q*-permutations, namely in case of combinatorial structures encoding orientable surfaces. The first one has been considered in [1] and consists in interpreting orientable transformations – *divide* and *merge* – as the effect of composing *q*-permutations with a suitable transposition. Being established such an algebraic correspondence, the solution comes straightforwardly by stressing very standard achievements in theory of permutations. The other solution provides a geometrical and interactive approach. For answering to our question '$\alpha \prec \beta$?' we compute the surface $\mathscr{S}_\alpha * \mathscr{S}_\beta$ obtained by composing, through identification of paired edges occurring on the boundaries, the two surfaces $\mathscr{S}_\alpha$ and $\mathscr{S}_\beta$ respectively corresponding to α and β. In [7], it is proved that the topological genus of $\mathscr{S}_\alpha * \mathscr{S}_\beta$ provides information enough to decide relaxation.

The passage from *q* to *pq*-permutations turns out to be critical from the point of view of the decision of relaxation. Whereas orientable transformations exclusively act at the level of the combinatorial structure of pq-permutations, *Möbius* and *Klein* also affect their supports so as to make impossible any resort to theory of permutations. On the contrary, we guess that the just mentioned geometrical procedure might admit a very natural extension in order to include non-orientable transformations and surfaces. As a matter of fact, it is an often remarked logical phenomenon that genuinely interactive approaches allow to avoid technical problems due to syntactical bureaucracy.

5 At the end...

Our logical system sPL generalizes Permutative logic, and in this way it is an embedding of the multiplicative Cyclic Linear logic (CyLL) and MLL. A completely unexplored field of research is that one of proof-nets for PL and sPL. The starting point should be the criterion for proof-nets of Planar Logic which just consists in requiring, together with the logical correctness, the planarity of the graph [5]. In this direction the main difficulty is that structural rules are usually 'transparent' with respect to the syntax of proof-nets so as we need to recover this kind of information by stressing the geometrical structure of the net. As far as PL is concerned, some useful results could be borrowed from [6], whereas the non-orientable side of the question misses at all of contributions.

Finally we developed a framework allowing to characterize the relationship between logic and orientable as well as not orientable surfaces. Semantical issues have not yet been explored and we are peculiarly interested in denotational semantic to give a topological interpretation of formulas and proofs. The existence of a phase semantics for sPL may provide some alternative algebraic tools for studying the geometry of 2-manifolds. A standard application of phase semantics consists in singling out redundant rules, namely rules which are superfluous with respect to the deductive power of the system, typically the cut rule. Now, since the basic topological transformations are embodied into our system, the classical classification theorem might find an interesting alternative proof when addressed in terms of semantics.

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